LECTURE SUMMARY SHEET FOR ANALYSIS SECTION OF MTH1A11. THESE ARE DESIGNED TO SUPPLEMENT YOUR LECTURE NOTES – NOT REPLACE THEM!

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Section 1: Numbers and sequences.

Objectives

(1) A real number is a *limit* of a *sequence* of finite decimal expressions.
(2) Two real numbers that are not equal differ by some *positive* distance and so must have different decimal expansions.
(3)The *modulus* $|\cdot|$ allows arguments about the distance between numbers to be written precisely.
(4) The modulus and *quantifiers* allow us to precisely define what it means for a sequence of numbers to converge.
(5) An increasing bounded sequence converges.

Analysis deals with the real numbers. In order to understand what they are, we start off by considering the following example. Let $f(x) = x^2 - 2$. Then the graph of f for x between 0 and 2 looks like it crosses the $y = 0$ axis (**Draw the graph**). On the other hand, we know that $\sqrt{2}$ is irrational, so if we do all our calculations over the rational numbers $\mathbb{Q}$ (which we – and computers – often do in practice), then the graph somehow leaps from the negative y side of the x–axis to the positive y side of the x–axis without ever crossing it (because $\sqrt{2}$ is irrational).

In order to do analysis, we need to be able to work with all the real numbers, not just the rationals. The positive square root of 2 is an example of a real number:

$$\sqrt{2} = 1.414213562... \qquad (1)$$

The problem is – what does that mean? And how could we use the infinite decimal expansion to prove results like $(1.414213562...)^2 = 2.000...$?

Definition 1.1. The real numbers are the numbers that may be written using infinite decimal expansions. Different expansions are allowed to represent the same real number (for example, $1.099999\ldots$ and 1.1 are the same real number).

It is very important to understand that the *representation* of a given real number as an infinite decimal is sometimes not unique. Think of the infinite decimals as a language for talking about the real numbers which has the small difficulty that some “words” (real numbers) can be correctly translated in two different ways.

Let’s prove that 1.1 and 1.0999... are different decimal representation for the same real number. Try to follow the two proofs very carefully: the first is a little complicated but only uses simple properties of *finite* decimals (that is, rational numbers), and manipulation of quantifiers and inequalities. This is a GOOD proof in that all of it can be described using things we understand well.

GOOD proof that $1.1000... = 1.09999...$ Assume the two decimal expansions represent different real numbers. Then either

$$1.1000... > 1.09999...$$

or

$$1.1000... < 1.09999...$$

Let's assume the first case (the second leads to a similar contradiction). If

$$1.1000... > 1.09999...$$

then

$$1.1000... - 1.09999... = \epsilon > 0$$

for some strictly positive ϵ.

Now consider the number $T = 1.099...9$ in which there are exactly n digits after the point. Since

$$1.0999... - T = 0.0...0999... > 0$$

we have that $1.0999... > T$ so

$$1.1000... - T > 1.1000... - 1.09999... = \epsilon.$$

On the other hand, $1.1000... - T$ is a difference of two finite decimals so we may compute it easily:

$$1.1 - T = 0.0...01 = 10^{-n}$$

(the 1 is in the nth place). Now let us choose n with the property that $10^{-n} < \frac{1}{10}\epsilon$. But this is impossible because the difference is supposed to be larger than ϵ.

This contradiction proves that $1.1000... > 1.09999...$ is impossible, so we deduce that $1.1000... \leq 1.09999...$.

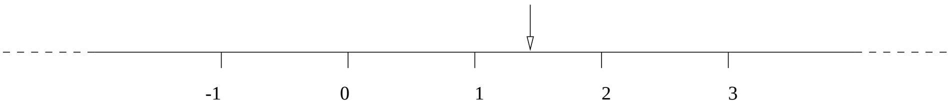
A similar proof shows that $1.1000... < 1.09999...$ is impossible, and we are done. □

For the second "proof", we will make some fairly vague assumptions about infinite decimals and how they can be manipulated. This is a BAD proof because it assumes we can treat infinite decimals in a certain way, when the whole point of the argument is to show that infinite decimals behave in a rather strange way. Follow this proof carefully and try to work out where we are doing something that cannot be justified using simple well–understood properties of finite decimals and inequalities.

BAD proof that $1.1000... = 1.09999...$ Let $S = 1.0999....$ Then[1] $10S = 10.999...$, so[2] $10S - S = 9S = 10.999... - 1.0999... = 9.9$, so dividing[3] by 9 we see that $S = 1.1$. $\square$

The moral is this: if we want to build up an understanding of real numbers (and hence analysis) from our sound basis of the rationals, finite decimals, inequalities, algebra and so on, then we must be prepared to use careful proofs by contradiction involving the existential quantifiers $\exists$ and $\forall$.

Now we have a better understanding of the non–uniqueness problem, let us make a formal interpretation of (1). Imagine the real numbers are points on a number line:



The real number $\sqrt{2}$ is the point indicated with an (infinitely thin) arrow. The equation (1) means that a certain infinite decimal represents the same point, but we cannot manipulate the infinite decimal – all we can manipulate are finite decimals. So what is the exact connection between the finite decimals

$$1, 1.4, 1.41, 1.414, ... \qquad (2)$$

and $\sqrt{2}$?

Definition 1.2. A list of numbers $a_1, a_2, a_3, \dots$ is called a *sequence*.

Sequence are sometimes written (a_n) or $\{a_n\}$.

So (1) says that $\sqrt{2}$ is represented by a sequence of finite decimals. However some sequence clearly do not represent real numbers: $1, 2, 3, 4, ...$ for example. The sequence (2) is not just an arbitrary sequence - it is a sequence getting closer and closer to some real number. For technical reasons, it is useful to restrict attention to *increasing sequences* (for example, if a_n is the nth decimal expansion of a number then (a_n) is an increasing sequence).

Definition 1.3. A sequence (a_n) is *increasing* if $a_n \leq a_{n+_1}$ for every $n \geq 1$. A sequence (a_n) is *bounded above* if there is some number R with $a_n \leq R$ for all $n \geq 1$.

Notice how we may use quantifiers to write these definitions:

(a_n) is increasing if $a_n \leq a_{n+1}$ $\forall$ $n \geq 1$;

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[1] A very dubious step: I am claiming that if the billionth digit of some decimal expansion of S is a 9 then there is a decimal expansion of $10S$ whose billionth digit is also a 9. If pressed I could prove this, but the proof would begin: assume not; then every decimal expansion of S has the property that the billionth digit is *not* a 9... and we'd end up with an even more complicated argument than the GOOD proof.

[2] More problems: how on earth do we perform an infinitely long subtraction problem?

[3] This step is now back in the world of finite decimals and so is okay.

(a_n) is bounded above if $\exists\ R$ such that $a_n \leq R\ \forall\ n \geq 1$.

How can we express the fact that the sequence (1) is getting closer and closer to $\sqrt{2}$? Our statement must involve potentially ALL the terms of the sequence:
1 is within distance 1 of $\sqrt{2}$ since they have the same integer part.
1.4 is within distance 0.1 of $\sqrt{2}$ since they have the same integer part and first one decimal digit.
1.41 is within distance 0.01 of $\sqrt{2}$ since they have the same integer part and the same first two decimal digits.
1.414 is within distance 0.001 of $\sqrt{2}$ since they have the same integer part and the same first three decimal digits.
and so on *ad infinitum*.

The reason why the sequence of decimals (1) is converging to $\sqrt{2}$ is not just that the terms get closer and closer to $\sqrt{2}$: they get *arbitrarily* close. That is, given any positive number at all ($\epsilon > 0$) the terms of the sequence are eventually (so long as we include at least $N = N(\epsilon)$ terms) distance less than that positive number from $\sqrt{2}$.

Recall that $|x|$ is the *absolute value* of x; $|-3| = 3$ and so on. It follows that $|x - y|$ is the *distance between* x and y. Thus, numbers x and y are close together if $|x - y|$ (which is ≥ 0) is *small*. The basic property of $|\ |$ that we need is the *triangle inequality*:

$$|x + y| \leq |x| + |y|. \tag{$\boldsymbol{\Delta}$}$$

We proved this in lectures – make sure you see why it is true as we will use it often. A useful corollary of $\boldsymbol{\Delta}$ (**prove this**) is the following:

$$|x + y| \geq |x| - |y|. \tag{$\boldsymbol{\Delta}'$}$$

Definition 1.4. A sequence (a_n) converges to the real number λ if, given any $\epsilon > 0$ there exists N such that $n \geq N$ implies that $|\lambda - a_n| < \epsilon$. The number λ is called the *limit* of the sequence, and we write

$$\lambda = \lim_{n\to\infty} a_n$$

or

$$a_n \longrightarrow \lambda.$$

Example 1.5. This examples shows how to use Definition 1.4 in practice.
Problem: Prove that the sequence defined by $a_n = 1 + \frac{1}{\sqrt{n}+2}$ converges.
Solution: An educated guess is that a_n will converge to 1. According to Definition 1.4, we must show that $|a_n - 1|$ can be made arbitrarily small by choosing n large enough.

First do a little side calculation:

$$|a_n - 1| = \left|\frac{1}{\sqrt{n} + 2}\right|,$$

so if I want to ensure that $|a_n - 1| < \epsilon$, I need to be sure that $n > \left(\frac{1}{\epsilon} - 2\right)^2$.

We are now ready to do the proof:
Given any $\epsilon > 0$, define $N = \left(\frac{1}{\epsilon} - 2\right)^2 + 1$. Then, for any $n \geq N$, we have

$$|a_n - 1| = \left|\frac{1}{\sqrt{n} + 2}\right| < \epsilon,$$

proving that $a_n \to 1$ as $n \to \infty$.

We have seen that the specific real number $\sqrt{2}$ is the *limit* of a specific increasing sequence of decimal approximations. More generally we have the following general result:

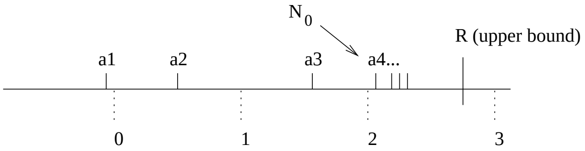
Theorem 1.6. *If (a_n) is an increasing sequence bounded above, then it converges: there is a $\lambda \in \mathbb{R}$ such that $a_n \longrightarrow \lambda$.*

This theorem may be thought of as telling us what the real numbers look like – *anything you can make appear as the limit of an increasing bounded sequence is a real number.* The method of proof is to simply start producing the decimal digits of the limit.

It also will allow us to exhibit subtle phenomena: for example, does the sequence $1, 1 + \frac{1}{2} - \ln 2, 1 + \frac{1}{2} + \frac{1}{3} - \ln 3, \ldots$ converge? If so, to what?

Proof. First decide what the integer part of the limit should be:

Let x be the largest integer with the property that eventually (for all n larger than some number N_0), all the terms a_n of the sequence are larger than or equal to x and smaller than $x + 1$.



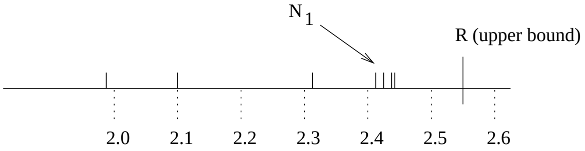
Here $x = 2$.

We now know that

$$n > N_0 \implies |x - a_n| \leq 1.$$

Next, simply enlarge the previous diagram by a factor of ten and decide what the first decimal digit should be:

Let x_1 be the largest digit (a whole number between 0 and 9) with the property that eventually (for all n larger than some number N_1), all the terms a_n of the sequence are larger than or equal to $x.x_1$ and smaller than $x.x_1 + 0.1$.



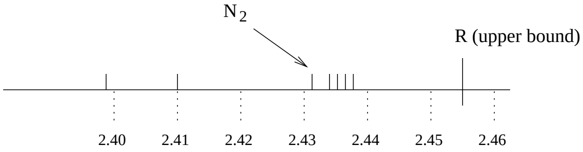
Here $x_1 = 4$.

We now know that

$$n > N_1 \implies |x.x_1 - a_n| \leq 0.1.$$

And so on: enlarge by a factor of ten again and decide what the second decimal digit should be:

Let x_2 be the largest digit (a whole number between 0 and 9) with the property that eventually (for all n larger than some number N_2), all the terms a_n of the sequence are larger than or equal to $x.x_1x_2$ and smaller than $x.x_1x_2 + 0.01$.



Here $x_2 = 3$.

We now know that

$$n > N_2 \implies |x.x_1x_2 - a_n| \leq 0.01.$$

Continuing, we produce the real number

$$\lambda = x.x_1x_2x_3x_4\ldots,$$

and numbers $N_0, N_1, N_2, \ldots$ with the property that

$$n > N_k \implies |a_n - \lambda| \leq 10^{-(k-1)}.$$

Given any $\epsilon > 0$, choosing k large enough to ensure that $10^{-(k-1)} < \epsilon$ produces N_k with the property that

$$n > N_k \implies |a_n - \lambda| < \epsilon.$$

Thus the sequence converges and the limit is the real number λ. $\square$

Remark 1.7. Notice that if x and y are real numbers, then either $x = y$ or there exists some $\epsilon > 0$ such that $|x - y| > \epsilon$.

Or, as Euler put it:

If a nonnegative quantity was so small that it is smaller than any given one, then it certainly could not be anything but zero. To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as there are usually believed to be. These supposed mysteries have rendered the calculus of the infinitely small quite suspect to many people. Those doubts that remain we shall thoroughly remove in the following pages, where we shall explain this calculus.

LEONHARD EULER (1707 - 1783)

A slightly more difficult idea is the following: by negating the quantified statement of convergence, we can prove that some sequences *do not* converge.

It is useful to record first a "reverse" triangle inequality: we know that

$$|x+y| \leq |x| + |y|$$

for all real numbers x and y. A little thought shows that

$$|x+y| \geq |x| - |y|$$

for all real numbers x and y.

Before the next example, let us recall how to *negate quantified statements.* Here is a quantified statement: in each year at UEA (first, second, third) there is a student all of whose relatives are tall. More formally:

$$\underbrace{\text{in each year group at UEA}}_{\forall\ j\in\{1,2,3\}}\ \underbrace{\text{there is a student}}_{\exists \text{ a student } s \text{ in year } j}\ \underbrace{\text{all of whose relations are tall.}}_{\forall\ r,\ r \text{ a relation of } s,\ r \text{ is tall}} \qquad (\dagger)$$

That is:

$$\forall\ j \in \{1,2,3\}\ \exists\ s \in \text{year } j \text{ such that } \forall\ r,\ r \text{ a relation of } s,\ r \text{ is tall.}$$

Notice that the last statement is an implication:

$$\forall\ j \in \{1,2,3\}\ \exists\ s \in \text{year } j \text{ such that } \forall\ r,\ r \text{ related to } s \implies r \text{ is tall,} \qquad (\ddagger)$$

where the last "for all" is taken over all people. We will negate the statement (‡). Firstly, the statement says

$$\forall\ j \in \{1,2,3\}, P(j)$$

where

$$P(j) \equiv (\exists\ s \in \text{year } j \text{ such that } \forall\ r,\ r \text{ related to } s \implies r \text{ is tall.})$$

So[4] the negation is

$$\exists\ j \in \{1,2,3\}, \text{not } P(j).$$

We now turn to the statement "not $P(j)$". Since $P(j)$ begins

$$\exists\ s \in \text{year } j \text{ such that } Q(s)$$

the negation is

$$\forall\ s \in \text{year } j, \text{ not } Q(s).$$

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[4] Notice that the negation of a *theorem* of the form **for all** $a \in A$**,** $Q(a)$ **holds** is a *counterexample*: **there exists** $a \in A$ **such that** $Q(a)$ **does not hold.**

So the negation of (‡) may be written:

$$\exists\ j \in \{1,2,3\},\ \text{such that } \forall\ s \in \text{year } j,\ \text{not } Q(s),$$

where

$$Q(s) \equiv (\forall\ r,\ r \text{ related to } s \implies r \text{ is tall,})$$

Now $Q(s)$ says that for all r a certain property (r being related to s) implies that r is tall. The negation is therefore a counterexample: an r which satisfies the property but is not tall:

$$\text{not } Q(s) \equiv (\exists\ r,\ r \text{ related to } s,\ \text{such that } r \text{ is short.})$$

Putting it all together, we have the negation of (‡):

$$\exists\ j \in \{1,2,3\},\ \text{such that } \forall\ s \in \text{year } j\ \exists\ r,\ r \text{ related to } s,\ \text{s.t. } r \text{ is short.} \qquad (\sharp)$$

That is, there is a year in which every student has a relative who is short. It may help to compare (♯) with (‡):

$$\forall\ j \in \{1,2,3\}\ \exists\ s \in \text{year } j \text{ such that } \forall\ r,\ r \text{ related to } s \implies r \text{ is tall,} \qquad (\ddagger)$$

to convince yourself that negation is not difficult!

Example 1.8. Write down a precise definition of the statement "(a_n) does not converge". Using this, prove that $a_n = (-1)^n$ does not converge.
Solution: Recall that the statement "(a_n) converges to λ" means:

$$\forall\ \epsilon > 0\ \exists\ N \text{ such that } n > N \implies |a_n - \lambda| < \epsilon.$$

It follows that the statement "(a_n) converges" means:

$$\exists\ \lambda,\ \forall\ \epsilon > 0\ \exists\ N \text{ such that } n > N \implies |a_n - \lambda| < \epsilon.$$

Negating this gives the statement "(a_n) does not converge":

$$\forall\ \lambda\ \exists\ \epsilon > 0 \text{ such that } \forall\ N\ \exists\ n > N \text{ with } |a_n - \lambda| \geq \epsilon. \qquad (3)$$

Notice how this works: in particular, the negation of the statement $n > N \implies |a_n - \lambda| < \epsilon$ is to say there is a counterexample: $\exists\ n > N$ with $|a_n - \lambda| \geq \epsilon$.

To solve the problem we must therefore prove that the sequence $a_n = (-1)^n$ has property (3). The idea is that if some a_n is very close to a number λ, then some other term in the sequence *cannot* be. Let λ be any number. Choose $\epsilon = \frac{1}{4}$. Let N be any natural number. Notice that for each "$\forall$" we have simply chosen an arbitrary number, and for the "$\exists$" we have had to think a bit and choose a specific number.

Now, look at a_N. If

$$|a_N - \lambda| \geq \tfrac{1}{4}$$

then (3) is true and we are done. If not, then

$$|a_N - \lambda| < \tfrac{1}{4},$$

which means that

$$a_N \in (\lambda - \tfrac{1}{4}, \lambda + \tfrac{1}{4}). \tag{4}$$

Assume that (4) holds, and recall that $|a_{n+1} - a_n| = 2$ for all n. By the "reverse" triangle inequality,

$$|a_{N+1} - \lambda| = |(a_{N+1} - a_N) + (a_N - \lambda)\,| \geq |a_{N+1} - a_N| - |a_N - \lambda| \geq 2 - \tfrac{1}{4} > \tfrac{1}{4}.$$

So either a_N is more than $\frac{1}{4}$ from λ or a_{N+1} is: we have proved property (3) and deduce that the sequence (a_n) does not converge.

Things you should have learned in this section

(1) Any sequence (a_n) of finite decimals, with the property that the first n digits of $a_n, a_{n+1}, \ldots$ agree, converges to a unique *real number*. Any real number can be written (possibly in more than one way) as an infinite decimal expansion.
(2) Exact definition of the statement "(a_n) converges to λ".
(3) How to prove using the definition that a given sequence converges.
(4) How to prove using the definition that a given sequence does *not* converge.
(5) If a sequence is *bounded* and *increasing*, then it is convergent.

TBW/1A11

Lecture summary sheet for Analysis section of MTH1A11.

Section 2: Convergent sequences.

Objectives

(1) Proofs of convergence and divergence of sequences.
(2) Algebra of convergent sequences.
(3) How to use increasing + bounded to show that some sequences converge without knowing the limit.

In this section we find how to manipulate convergent sequences and do some more examples.

Examples 2.1. Consider the following four examples.
(1) $\{\frac{1}{n}\}$, the sequence $c_n = \frac{1}{n}$, i.e. $1, \frac{1}{2}, \frac{1}{3}, \ldots$.
(2) $\{\sqrt{n+1} - \sqrt{n}\}$, the sequence $d_n = \sqrt{n+1} - \sqrt{n}$, i.e. $\sqrt{2} - 1, \sqrt{3} - \sqrt{2}, \ldots$.
(3) $\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\}$.

We make the following observations:
(1) the sequence $\{c_n\}$ gets closer and closer to zero.
(2) we shall see that $\{d_n\}$ also approaches zero.

Recall that a sequence (a_n) *converges* to ℓ if for any $\epsilon > 0$ there is an N with the property that $n > N$ implies $|a_n - \ell| < \epsilon$.

Notation 2.2. A sequence that does not converge to any ℓ is said to *diverge*. For a convergent sequence, the number ℓ is called the *limit* of the sequence.

Example 2.3. The sequence $\{c_n\} = \{\frac{1}{n}\}$ converges to zero. In order to prove this statement, we must establish that Definition 1.4 holds for this sequence with $\ell = 0$. Given any $\epsilon > 0$, choose N to be the smallest natural number with the property that $N > \frac{1}{\epsilon}$. Then, if $n > N$, we have $|c_n - 0| = \frac{1}{n} < \frac{1}{N} < \epsilon$, which proves that $\{c_n\}$ is converging to zero.

Example 2.4. The sequence $\{d_n\} = \{\sqrt{n+1} - \sqrt{n}\}$ converges to zero. This is not so obvious, and we should first convince ourselves that it is true – once we know why it is true we should be able to prove it. For this, we need a trick so useful that it has a name, "difference of two squares":

$$a^2 - b^2 = (a-b)(a+b), \tag{$\blacksquare - \blacksquare$}$$

which becomes (with $a = \sqrt{n+1}$ and $b = \sqrt{n}$)

$$1 = (\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n}).$$

This says that $d_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$, which is clearly converging to zero since $\sqrt{n+1} + \sqrt{n}$ is going off to infinity. Now we're ready for the proof. Given $\epsilon > 0$, choose N so that $N > \frac{1}{4\epsilon^2}$. Then, if $n > N$, $|d_n - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{2\sqrt{N}} < \epsilon$.

Theorem 2.5. *If $a_n \to \ell$ and $b_n \to m$, then*
(1) $(a_n + b_n) \to \ell + m$ (equivalently, $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ if both limits on the right exist).
(2) $a_n b_n \to \ell m$ (equivalently, $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} b_n$ if both limits on the right exist).

Discussion. Proof of (1): given $\epsilon > 0$, we know that there are numbers N_1 and N_2 with the property that

$$n > N_1 \implies |a_n - \ell| < \frac{\epsilon}{2} \quad \textit{and} \quad n > N_2 \implies |b_n - m| < \frac{\epsilon}{2}.$$

So if $N = \max\{N_1, N_2\}$, then

$$\begin{aligned} n > N \implies |(a_n + b_n) - (\ell + m)| &= |a_n - \ell + b_n - m| \\ &\le |a_n - \ell| + |b_n - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $a_n + b_n \to \ell + m$.
Proof of (2). In order to prove (2), we must show that the quantity $|a_n b_n - \ell m|$ is eventually small. By $\boldsymbol{\Delta}$, $|a_n b_n - \ell m| = |(a_n - \ell)b_n + \ell(b_n - m)| \le |b_n| \cdot |a_n - \ell| + |\ell| \cdot |b_n - m|$, which will be fine so long as we know that $|b_n|$ is bounded. For this we need a

Useful Lemma 2.6. *If $\{b_n\}$ is a convergent sequence, then it is bounded: there is a number R such that $|b_n| \le R$ for all n.*

Proof of Useful Lemma. We have $b_n \to m$; take $\epsilon = 1$ in Definition 1.4. Then there is a number M such that $n > M \implies |b_n - m| < 1$. Now if $|b_n - m| < 1$, then b_n is within a distance 1 of m, so that $|b_n| < 1 + |m|$. It follows that

$$|b_n| \le \max\{|b_1|, |b_2|, \ldots, |b_M|, 1 + |m|\},$$

which proves the Lemma. ☐

Now we may continue with the proof of Theorem 2.5 part (2). By the Useful Lemma, we may find a number R with two properties: $R > |b_n|$ for all n, and $R > |\ell|$. Now $a_n \to \ell$ and $b_n \to m$, so given any $\epsilon > 0$ we may choose N_1 and N_2 such that

$$n > N_1 \implies |a_n - \ell| < \frac{\epsilon}{2R} \quad \text{and} \quad n > N_2 \implies |b_n - m| < \frac{\epsilon}{2R}.$$

Then if $N = \max\{N_1, N_2\}$,

$$\begin{aligned} n > N \implies |a_n b_n - \ell m| &\le |b_n| \cdot |a_n - \ell| + |\ell| \cdot |b_n - m| \\ &< |b_n| \frac{\epsilon}{2R} + |\ell| \frac{\epsilon}{2R} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which proves that $a_n b_n \to \ell m$. ☐

A similar result holds for quotients of sequences, though we must avoid dividing by zero.

Theorem 2.7. *If $a_n \to \ell$, and $\{b_n\}$ is a sequence of non–zero terms converging to $m \neq 0$, then $\frac{a_n}{b_n} \to \frac{\ell}{m}$.*

Discussion. This was proved in the lectures. Notice that a key element of the proof was a version of the Useful Lemma above, showing that the sequence b_n is *bounded away from zero*: there is an $r > 0$ such that $r < |b_n|$ for all n. (One way to prove this is to consider the sequence $\{\frac{1}{b_n}\}$ – though to be sure this sequence is convergent you really need Theorem 2.7...) □

These two theorems allow more complicated limits to be found.

Example 2.8. Let $a_n = \frac{3n^3+7n^2+1}{8n^3-2n^2+n+1}$. Show that $\lim\limits_{n\to\infty} a_n = \dfrac{3}{8}$. First notice that $\frac{1}{n} \to 0$, $\frac{1}{n^2} \to 0$ and so on. Now

$$a_n = \frac{3 + \frac{7}{n} + \frac{1}{n^3}}{8 - \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n^3}} = \frac{b_n}{c_n},$$

and the numerator and denominator are both convergent sequences (by Theorem 2.5), so Theorem 2.7 applies:

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} b_n}{\lim_{n\to\infty} c_n} = \frac{3}{8}.$$

Recall Theorem 1.6: a bounded increasing sequence converges. Similarly, a bounded below decreasing sequence converges.

Example 2.9. Theorem 1.6 (and the non–increasing version) allow us to do some quite clever things. Here we use it to describe Euler's number, and to motivate some later material. Consider the function $f(x) = \log(x)$ (natural logs of course). We know that for $n \geq 1$, $I = \int_n^{n+1} \frac{1}{x}dx = \log(n+1) - \log(n)$. On the other hand, for $x \in [n, n+1]$, we have that $\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}$ (**Why?**). It follows that $\frac{1}{n+1} < I < \frac{1}{n}$ (**Why?** – see Lemma 3.16 below), so

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}.$$

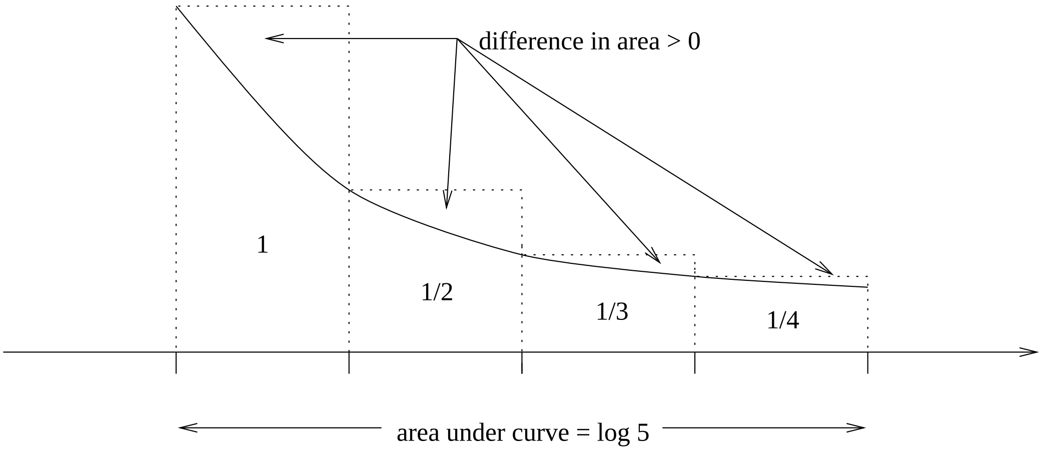
Consider the sequence

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n.$$

Now

$$\begin{aligned} a_{n+1} - a_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1} - \log(n+1)\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \\ &= \log(n) - \log(n+1) + \frac{1}{n+1} < 0 \end{aligned}$$

by the above calculation. It follows that the sequence $\{a_n\}$ is decreasing. Also, each a_n is greater than or equal to zero (**Why?** Hint: show that a_n may be represented as the area between a staircase function lying above the graph of $y = \frac{1}{x}$ and the curve itself. This picture should convince you that $a_n - \frac{1}{n}$ is clearly positive, which immediately shows that a_n is also). The picture below indicates why $a_5 - \frac{1}{5} > 0$, from which we see that $a_5 > 0$.



It follows that $\{a_n\}$ converges to some number γ: $a_n \to \gamma$. This number γ (approximately $0.57721566\ldots$) is called Euler's number and it is somewhat mysterious: in particular, it is not known whether or not it is rational.

Example 2.10. *[This example was not covered in the lectures]* The idea used above of estimating things by areas (which are in turn computed as definite integrals) gives some insight into the following difficult question: how big is $n!$? It is easy enough to check that $n!$ grows faster than any power of n: for any $k \in \mathbb{N}$, $\lim_{n\to\infty} \frac{n^k}{n!} = 0$. (**Prove this**). What about n^n? A convenient way to formulate this question is: if $a_n = \frac{\sqrt[n]{n!}}{n}$, then does $\{a_n\}$ converge to some number in $(0, \infty)$?

By drawing a graph, show that

$$\log 1 + \log 2 + \cdots + \log(n-1) < \int_1^n \log(x)dx < \log 2 + \log 3 + \cdots + \log(n).$$

Evaluate the integral, and apply the function $x \mapsto e^x$. Notice that (for example) $e^{\log 2 + \log 3 + \cdots + \log(n)} = 2 \cdot 3 \cdots n = n!$ and so on. Deduce the inequality:

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

Show that this implies that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e} \doteq 0.36788\ldots$$

What have we proved? The limit result shows that $\sqrt[n]{n!}$ is approximately $\frac{n}{e}$ (i.e. the ratio of the two quantities is close to 1) when n is large. This does not imply

(**Why not?**) that $n!$ is approximately $\left(\frac{n}{e}\right)^n$ in the same sense. An estimate for $n!$ that has the property that the ratio $\frac{n!}{\text{the estimate for } n!}$ converges to 1 exists but is quite difficult to prove. Here's a common way of writing the result (called Stirling's formula):

$$\sqrt{2\pi}\; n^{n+\frac{1}{2}} e^{-n} < n! < \sqrt{2\pi}\; n^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}.$$

Things you should have learned in this section

(1) How to decide if a sequence converges, and how to write down a proof.

(2) How to use the algebra of convergent sequences to prove that more complicated sequences converge.

(3) How to use various methods (algebraic tricks, representation as integrals) to prove that sequences converge.

(4) All convergent sequences are bounded, and how to prove this.

TBW/1A11

Lecture summary sheet for Analysis section of MTH1A11.

Section 3: Series.

Objectives

(1) An understanding of infinite sums and when they converge.
(2) How to use our knowledge of *sequences* to prove things about *series*.
(3) Test for convergence of series.

A *series* is an infinite sum. We have already encountered and used series without noticing the fact. We used real numbers like $\sqrt{2} = 1.4141....$ This number is really an infinite sum:

$$\sqrt{2} = 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} + \dots \tag{*}$$

What does (*) really mean? Does it mean that any calculation involving $\sqrt{2}$ involves infinitely many additions? Section 2 gives us a good idea of how to interpret (*): it means that we may approximate $\sqrt{2}$ as well as we like by taking sufficiently many terms on the right of (*). This is just the observation that the decimal expansion of a real number out to n places is a close approximation to the number, how close depending on n. This idea is so often used that you don't notice it when you use it: every calculation you have done with π for example, has really been done with some finite piece of the decimal expansion, 3.1415926 say. More generally, a *series* is going to be an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots \tag{S}$$

(the Greek letter Σ represents a capital S for "sum"). By analogy with the properties of real numbers discussed above, we shall say that the series (S) *converges to* ℓ if given any positive distance $\epsilon > 0$ the finite sums of the form $a_1 + a_2 + \cdots + a_n$ are within ϵ of the fixed number ℓ for n big enough (how big depending on ϵ).

Provisional Definition 3.1. The series $\sum_{n=1}^{\infty} a_n$ converges to ℓ, written symbolically $\sum_{n=1}^{\infty} a_n = \ell$, if given any $\epsilon > 0$ there is an M such that if $N > M$ then $\left|\sum_{n=1}^{N} a_n - \ell\right| < \epsilon$.

Example 3.2. Let $a_n = \frac{1}{2^n}$. Then $\sum_{n=1}^{\infty} a_n = 1$. To prove this, simply evaluate $\sum_{n=1}^{N} a_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N} = 1 - 2^{-N}$. It is clear that $2^{-N} \to 0$ as $N \to \infty$, so by applying Section 2 we see that $\sum_{n=1}^{\infty} a_n = 1$.

The argument used in the above example points the way to a more useful formulation of convergence for series. Given the series $\sum_{n=1}^{\infty} a_n$, form the associated *sequence of partial sums* $\{s_m\}$, defined by $s_m = \sum_{n=1}^{m} a_n$.

Definition 3.3. The series $\sum_{n=1}^{\infty} a_n$ converges to ℓ if and only if the sequence of partial sums $\{s_m\}$ converges (as a sequence) to the number ℓ.

Logical Aside

Before discussing tests for convergence of series, it is useful to recall the *converse* and *contrapositive* of a logical statement. We are interested in statements of the form:

$$A: \quad p \implies q.$$

The *converse* statement is

$$con(A): \quad q \implies p,$$

and the *contrapositive* statement is "not q implies not p":

$$\neg q \implies \neg p.$$

For example, if

$$p = \text{"you are in a running shower"}$$

and

$$q = \text{"you are wet"}$$

then $p \implies q$, "if you are in a running shower, then you are wet" is `TRUE`. The contrapositive statement "if you are dry, then you cannot be in a running shower" is also `TRUE`. If you think about it, this is always going to be the case: a statement and its contrapositive are either both `TRUE` or both `FALSE`.

On the other hand, the converse of statement A is the statement "if you are wet, then you must be in a running shower". This is clearly `FALSE`. As usual, we show that something is false by exhibiting a *counterexample:* you could be in a swimming pool.

A convergent series will also be called *summable* for obvious reasons. In this section we want to develop tools for deciding whether or not a series is summable. The simplest of all tests is the following.

Lemma 3.4. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Discussion. Let s_m denote the mth partial sum of the series. Then, if $\sum_{n=1}^{\infty} a_n = \ell$, we have that $s_m \to \ell$ as $m \to \infty$. But then

$$lim_{m\to\infty} a_m = \lim_{m\to\infty} (s_m - s_{m-1}) = \lim_{m\to\infty} s_m - \lim_{m\to\infty} s_{m-1} = \ell - \ell = 0.$$

Be warned: this result only works in one direction: we shall see in the next example that $a_n \to 0$ does not imply that the series is summable. ☐

Example 3.5. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not summable. This is proved by a simple grouping argument which shows that the sequence of partial sums is not bounded (and therefore is not convergent by the Useful Lemma of Section 2). Notice that $\frac{1}{3} + \frac{1}{4} \geq \frac{1}{2}$ (sum of two terms each $\geq \frac{1}{4}$). Similarly, $\frac{1}{5} + \cdots + \frac{1}{8} \geq \frac{1}{2}$ (sum of 4 terms each $\geq \frac{1}{8}$). Similarly, $\frac{1}{9} + \cdots + \frac{1}{16} \geq \frac{1}{2}$ (sum of 8 terms each $\geq \frac{1}{16}$). That is, if we groups the terms into blocks of length $1, 1, 2, 4, 8, \ldots$ we arrive at the following picture:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{17} + \cdots + \frac{1}{32}}_{\geq \frac{1}{2}} + \ldots$$

It follows that if $n = 1 + 1 + 2 + 4 + 8 + \cdots + 2^k$, then $s_n \geq \frac{k}{2}$, so $\{s_n\}$ is not bounded, and therefore not convergent, so the series is not summable.

Subtle aside. The above discussion may be pushed a little further: the harmonic series diverges, but just how fast or slowly does this happen? An approximate answer is the following: if $n = 1 + 1 + 2 + 4 + 8 + \cdots + 2^k \doteq 2^{k+1}$, then $s_n \doteq k$ (the odd $+1$ or $\times \frac{1}{2}$ does not matter here). Put another way: for certain values of n, s_n is approximately $\log(n)$. We already knew that – and knew it very precisely – from Example 2.11. In that example, we showed that s_n differs from $\log(n)$ by an amount that is converging as $n \to \infty$.

Example 3.6. The geometric series $\sum_{n=1}^{\infty} r^n$ is summable if $|r| < 1$ and is not summable if $|r| \geq 1$. This is clear: for $|r| \neq 1$ the mth partial sum is simply given by $s_m = \frac{1 - r^{m+1}}{1 - r}$.

What about series like $\sum \frac{1}{n^2}$, or $\sum \frac{2 + \sin(n^2 + 3)}{2^n + n^3}$? For series with non–negative terms, the next Lemma is very useful.

Lemma 3.7. *If all the terms a_n are non–negative, then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the set of partial sums is bounded.*

Discussion. This follows from Theorem 1.6: if the terms are non–negative, then the sequence of partial sums is non–decreasing. (Series with some positive and some negative terms may fail to converge even if the sequence of partial sums is bounded: an example is Example 3.19 below). ☐

Theorem 3.8. [THE COMPARISON TEST] *Suppose that $0 \leq a_n \leq b_n$ for all n. Then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.*

Discussion. Let $s_m = a_1 + \cdots + a_m$ and $t_m = b_1 + \cdots + b_m$. Then for all m, $0 \leq s_m \leq t_m$. Now $\{t_m\}$ is bounded, since $\sum_{n=1}^{\infty} b_n$ converges. It follows that $\{s_m\}$ is bounded, so $\sum_{n=1}^{\infty} a_n$ converges too. ☐

Example 3.9. The series $\sum_{n=1}^{\infty} \frac{2+\sin(n^2+3)}{2^n+n^3}$ converges. This is easy:

$$0 \leq \frac{2+\sin(n^2+3)}{2^n+n^3} \leq \frac{3}{2^n},$$

and we know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

A more sophisticated version of Theorem 3.8 is the following result:

Theorem 3.10. [THE LIMIT COMPARISON TEST] *Suppose that* $a_n, b_n > 0$ *for all* n*, and* $\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0$*. Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} b_n$ *converges.*

Discussion. Suppose $\sum_{n=1}^{\infty} b_n$ converges. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, there is some N such that $a_n \leq 2cb_n$ for all $n \geq N$. But the sequence $2c\sum_{n=N}^{\infty} b_n$ certainly converges. So by Theorem 3.8, $\sum_{n=N}^{\infty} a_n$ converges, which implies that $\sum_{n=1}^{\infty} a_n$ converges (**Why?**).

The converse is immediate since $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{c}$. □

Example 3.11. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{2^n+3-\sin^2(n^3)}$ converges. The solution is simple: let $a_n = \frac{1}{2^n+3-\sin^2(n^3)}$, $b_n = \frac{1}{2^n}$, and notice that $\frac{a_n}{b_n} \to 1$. We know that $\sum b_n$ converges, so by the Limit Comparison Test, so does $\sum a_n$.

Example 3.12. Prove that the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$ does not converge. Again, this follows from the Limit Comparison Test: the ratio $\left(\frac{n+1}{n^2+1}\right) / \left(\frac{1}{n}\right)$ converges to 1, and we know that the harmonic series does not converge.

Example 3.6 may be extended to give a new test that works for certain series that "look like" geometric series.

Theorem 3.13. [THE RATIO TEST] *Let* $a_n > 0$ *for all* n*, and suppose that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r.$$

Then if $r < 1$, $\sum_{n=1}^{\infty} a_n$ *converges. On the other hand, if* $r > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *does not converge.*

Discussion. If $r > 1$ then the terms a_n do not even approach zero, so there is no chance of the series converging.

If $r < 1$, then we may find a number s with $r < s < 1$. The hypothesis is that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

so we may find some N such that

$$n \geq N \implies \frac{a_{n+1}}{a_n} \leq s.$$

We can write this out in a list:

$$\begin{aligned}
a_{N+1} &\leq s a_N \\
a_{N+2} &\leq s a_{N+1} \leq s^2 a_N \\
&\vdots \\
a_{N+k} &\leq s^k a_N.
\end{aligned}$$

Now we know that $\sum_{k=0}^{\infty} a_N s^k = a_N \sum_{k=1}^{\infty} s^k$ converges since $s < 1$. So by the Comparison Test,

$$\sum_{n=N}^{\infty} a_n = \sum_{k=0}^{\infty} a_{N+k}$$

also converges. This implies that $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Example 3.14. Show that $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. Apply the Ratio Test with $a_n = \frac{1}{n!}$. The ratio is $\frac{a_{n+1}}{a_n} = \frac{1}{n+1} \to 0$, so the series converges.

Despite all of the above tests, we have still not been able to decide whether or not $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. (**Why does the ratio test not help?**). In fact an annoyingly large number of series will turn out to have limiting ratio $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ equal to 1, which is exactly where The Ratio Test tells us nothing! In Example 2.11 we have already seen a method to connect sums of sequence with areas under curves, and this is the key idea we need to develop the next convergence test.

Recall that *improper integrals* are defined as follows: if $f : \mathbb{R} \to \mathbb{R}$ is a function, we define the integral $\int_1^{\infty} f(x)dx$ to be the limit of the sequence $\{a_n\}$ with terms defined by $a_N = \int_1^N f(x)dx$, provided this limit exists. If the sequence does not converge, then we say that the improper definite integral does not exist (or diverges). In symbols,

$$\int_1^{\infty} f(x)dx = \lim_{N\to\infty} \int_1^N f(x)dx \quad \text{if the limit exists.}$$

Example 3.15. Let $f(x) = \frac{1}{x^p}$, where $p > 0$. Then $\int_0^{\infty} f(x)dx$ exists if and only if $p > 1$. To see this, calculate the terms of the sequence, $a_N = \int_1^N f(x)dx$. We find that

$$a_N = \int_1^N f(x)dx = \begin{cases} -\frac{1}{(p-1)} \cdot \frac{1}{N^{p-1}} + \frac{1}{p-1}, & p \neq 1 \\ \log N & p = 1 \end{cases}$$

from which it is clear that $\{a_N\}$ converges if and only if $p > 1$.

On the other hand, definite integrals of the (proper) form $\int_a^b f(x)dx$ may be interpreted as areas under curves, leading to the following simple but important result (which will be proved in another section of the course; for our purposes we will take it as being “obvious”. The only sense in which it is not obvious is that we have not really defined integration...).

Lemma 3.16. *If* $m \leq f(x) \leq M$ *for all* x *with* $a \leq x \leq b$*, then*

$$(b-a)m \leq \int_a^b f(x)dx \leq (b-a)M.$$

Discussion. If we agree to think of definite integrals as areas, draw a picture to convince yourself that this result must be true. □

Let us agree to call a function f *decreasing on* $[0,\infty)$ if $x > y \implies f(x) < f(y)$ for all x and y in $[0,\infty)$.

Theorem 3.17. *Suppose that* f *is positive and decreasing on* $[1,\infty)$*, and that* $f(n) = a_n$ *for all* n*. Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if the limit* $\lim_{N\to\infty} \int_1^N f(x)dx$ *(also written* $\int_1^\infty f(x)dx$*) exists.*

Proof. Notice first that saying $\lim_{N\to\infty} \int_1^N f(x)dx$ exists is exactly the same as saying that the series

$$\int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx + \ldots$$

converges. Since f is decreasing (**Check this by drawing a graph.**), we have

$$f(n+1) < \int_n^{n+1} f(x)dx < f(n).$$

The first half of this inequality shows that the series $\sum_{n=1}^{\infty} a_{n+1}$ may be compared to the series $\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx$, which shows that $\sum_{n=1}^{\infty} a_{n+1}$ (and hence $\sum_{n=1}^{\infty} a_n$) converges if $\lim_{N\to\infty} \int_1^N f(x)dx$ exists.

The second half of the inequality shows that the series $\sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx$ may be compared to the series $\sum_{n=1}^{\infty} a_n$, proving that $\lim_{N\to\infty} \int_1^N f(x)dx$ must converge if $\sum_{n=1}^{\infty} a_n$ converges. □

Example 3.18. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. This follows at once from the integral test and Example 3.15 above.

Example 3.18 gives us another proof that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges – this is in fact the *third* proof we have. (**Where is the third proof?**) It also, finally, tells us that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In the second year complex analysis course, you will find that this sum may be evaluated exactly: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Remark. The series we have considered have all terms of the same sign. Series in which terms can be both positive and negative are slippery customers. There is one simple general result, Leibnitz's Test, which is given below as Theorem 3.20, that shows the *alternating harmonic series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges.

Theorem 3.19. [LIEBNITZ'S TEST] *Let $\sum_{n=1}^{\infty} a_n$ be a series whose terms are alternately positive and negative. Suppose that the terms decrease in size (i.e. $|a_n| \geq |a_{n+1}|$ for all n) and converge to zero in size (i.e. $a_n \to 0$ as $n \to \infty$). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent.*

Discussion. The theorem is proved by the following argument. Let $s_n = a_1 + \cdots + a_n$ denote the nth partial sum of the series. First it is shown that

$$s_2 \leq s_4 \leq s_6 \leq \dots$$

and

$$s_1 \geq s_3 \geq s_5 \geq \dots$$

and finally

$$s_k \leq s_\ell$$

whenever k is even and ℓ is odd.

It follows that $\{s_{2n}\}$ is a non–decreasing sequence, bounded above by s_1, hence convergent. Let $\alpha = \lim_{n\to\infty} s_{2n}$.

Similarly, $\{s_{2n+1}\}$ is a non–increasing sequence, bounded below by s_2, hence convergent. Let $\beta = \lim_{n\to\infty} s_{2n+1}$.

We are nearly done: since $s_{2n+1} - s_{2n} = a_{2n+1} \to 0$, we have that $\alpha = \beta$, so the series is convergent. (**Why does $\alpha = \beta$ prove the series is convergent?**). ☐

Example 3.20. Let $a_n = \frac{(-1)^{n+1}}{n}$. Then $\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \dots$ converges by Leibnitz's test. As in other cases, more advanced methods from complex analysis allow the sum to be evaluated exactly: $\sum a_n = \log 2$.

Things you should have learned in this section

(1) The definition of convergence for series.

(2) Definition of convergence in terms of the sequence of partial sums, and consequences.

(3) Examples of series: harmonic, geometric, power series.

(4) Tests for convergence: comparison, limit comparison, ratio, and integral tests.